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Direct integration of generalized Lie symmetries of nonlinear Hamiltonian systems with two degrees of freedom: integrability and separability

M Lakshmanan and M Senthil Velan

Centre for Nonlinear Dynamics, Department of Physics, Bharathidasan University, Tiruchirapalli 620 024, India

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Abstract. Many of the integrable coupled nonlinear oscillator systems are associated with generalized Lie symmetries involving velocity dependent terms. For a class of systems with two degrees of freedom, we show that by integrating the characteristic equation associated with the generalized symmetries, the required involutive integrals of motion can be obtained explicitly in a straightforward manner, almost by inspection and without recourse to Noether's theorem. Further, all the separable coordinates can be obtained by integrating a subset of the characteristic equation associated with the coordinate variables alone. Our explicit examples include the two coupled generalized Henon-Heiles, quartic, sextic and other polynomial oscillator systems as well as the perturbed Kepler system.

1. Introduction

An important method of identifying nonlinear integrable dynamical systems with finite degrees of freedom is through an analysis of the associated invariance properties under the one-parameter Lie group of continuous transformations [1-3]. However, the standard Lie point symmetries for most coupled nonlinear oscillators are either trivial or insufficient to establish integrability. One needs to consider more general Lie symmetries involving velocity dependent terms, in order to capture the full invariance properties [4-6]. Considering nonlinear dynamical systems with two degrees of freedom corresponding to the Lagrangian

$$L=(1/2)(\dot{x}^2+\dot{y}^2)-V(x,y) \tag{1.1}$$

and the Hamiltonian

$$H=(1/2)(p_x^2+p_y^2)+V(x,y) \tag{1.2}$$

the equations of motion

$$\ddot{x}=\frac{\partial L}{\partial x}=\alpha_1 \qquad \ddot{y}=\frac{\partial L}{\partial y}=\alpha_2 \tag{1.3}$$

can admit generalized Lie symmetries in the 5D space $(t, x, y, \dot{x}, \dot{y})$ which leave (1.3) invariant. The infinitesimal form of such symmetries may be written as

$$\begin{aligned}
&t\to T=t+\varepsilon\xi(t,x,y,\dot{x},\dot{y})\\
&x\to X=x+\varepsilon\eta_1(t,x,y,\dot{x},\dot{y})\\
&y\to Y=y+\varepsilon\eta_2(t,x,y,\dot{x},\dot{y})\\
&\dot{x}\to \dot{X}=\dot{x}+\varepsilon(\dot{\eta}_1-\dot{x}\dot{\xi})\\
&\dot{y}\to \dot{Y}=\dot{y}+\varepsilon(\dot{\eta}_2-\dot{y}\dot{\xi}) \qquad \varepsilon\ll 1.
\end{aligned} \tag{1.4}$$

The resulting invariance conditions are [7]

$$\ddot{\eta}_1 - \dot{x}\ddot{\xi} - 2\dot{\xi}\alpha_1 = E\{\alpha_1\} \tag{1.5a}$$

$$\ddot{\eta}_2 - \dot{y}\ddot{\xi} - 2\dot{\xi}\alpha_2 = E\{\alpha_2\} \tag{1.5b}$$

where the infinitesimal generator of the group is

$$E = \xi\frac{\partial}{\partial t} + \eta_1\frac{\partial}{\partial x} + \eta_2\frac{\partial}{\partial y} + (\dot{\eta}_1 - \dot{x}\dot{\xi})\frac{\partial}{\partial \dot{x}} + (\dot{\eta}_2 - \dot{y}\dot{\xi})\frac{\partial}{\partial \dot{y}}. \tag{1.6}$$

Any solution (ξ, η_1, η_2) of (1.5) forms an admissible set of infinitesimal symmetries of (1.3). Methods to find the forms of the infinitesimal symmetries of typical systems with two degrees of freedom and locating the integrable choices have been discussed in detail elsewhere [4, 8, 9].

After obtaining the infinitesimal symmetries, to prove the integrability of the given dynamical system one has to find the required involutive integrals of motion from the symmetries, provided they exist. If a given set of symmetries is of Noether's type then it can lead to an integral of motion via Noether's theorem of the form [7, 10, 11]

$$I = (\xi\dot{x} - \eta_1)\frac{\partial L}{\partial \dot{x}} + (\xi\dot{y} - \eta_2)\frac{\partial L}{\partial \dot{y}} + f \tag{1.7}$$

where f is a function of (x, y, t) to be determined.

However, in this article we wish to show that the generalized Lie symmetries can be used directly to obtain the required information about the integrability and separability properties, without recourse to Noether's theorem, via specific examples by solving the first-order linear partial differential equation $E\{U\} = 0$ in five independent variables, where U is the local, group, invariant of the one-parameter group of transformations. For this purpose we consider the characteristic equation associated with the symmetry group (1.6). Out of the four group invariants or essential constants admitted by the general solution of $E\{U\} = 0$, two invariants turn out to be nothing but the two required involutive integrals of motion of (1.3) to prove its complete integrability which can be written down almost by inspection. The explicit determination of the remaining two group invariants (which are not connected to the integrals of motion) depends on the nature of the symmetry transformation (1.4). However, in all our examples they can also be determined in principle straightforwardly. Further, separable coordinates, whenever they exist, follow by integrating a subset of the characteristic equation associated with the coordinate variables only. We find that whenever the symmetry is linear in velocities, the associated subset of the characteristic equation corresponding to the coordinate variables becomes a solvable ordinary differential equation, while for symmetries which are of higher degree in velocities the associated subset becomes non-solvable explicitly.

In table 1 we summarize the non-trivial generalized symmetries associated with the following coupled nonlinear oscillators along with the specific parametric values for which the system becomes integrable along with the integrals of motion which will be derived by this method. The symmetries of examples (1)–(3) are given in [4, 8, 9] and for the remaining cases they are obtained by following the same method as given in these references:

(1) Coupled quartic oscillator [12]:

$$L = (1/2)(\dot{x}^2 + \dot{y}^2) - [Ax^2 + By^2 + \alpha x^4 + \beta y^4 + \delta x^2 y^2]. \tag{1.8}$$

Table 1.

S No	System	Cases	Parametric restrictions	Infinitesimal symmetries ξ	η_1	η_2	Integral of motion I_2
1	Coupled quartic anharmonic oscillator equation (1.8)	1	A, B arbitrary, $\alpha=\beta$, $\delta=2\alpha$	0	$2yL_{yx}+\frac{2}{\alpha}(B-A)\dot{x}$	$2xL_{xy}$	$L_{xy}^2+\frac{2}{\alpha}(B-A)\left[\frac{\dot{x}^2}{2}+Ax^2+\alpha\rho^2x^2\right]$
		2	$A=B$, $\alpha=\beta$, $\delta=6\beta$	0	$k\dot{y}$	$k\dot{x}$	$\dot{x}\dot{y}+2[A+2\alpha(x^2+y^2)]xy$
		3	$A=4B$, $\alpha=16\beta$, $\delta=12\beta$	0	$y\dot{y}$	$y\dot{x}-2x\dot{y}$	$\dot{y}L_{yx}+2[B+2B\rho^2+2\beta x^2]xy^2$
		4	$A=4B$, $\alpha=8\beta$, $\delta=6\beta$	0	$8\beta(y\dot{x}-2x\dot{y})y^3$	$4\dot{y}^3+8[B+\beta\rho^2+5\beta x^2]y^2\dot{y}-16\beta xy^3\dot{x}$	$\dot{y}^4+4[B+\beta\rho^2+5\beta x^2]y^2\dot{y}^2-16\beta xy^3\dot{x}\dot{y}+4\beta y^4\dot{x}^2+4B(B+2\beta\rho^2+2\beta x^2]y^4+4\beta^2(\rho^2+x^2)^2y^4$
2	Henon–Heiles equation (1.9)	1	$A=B$, $C=-D$	0	$k\dot{y}$	$k\dot{x}$	$\dot{x}\dot{y}+Axy-\frac{Cx^3}{3}-Cxy^2$
		2	A, B arbitrary, $C=-6D$	0	$4D(x\dot{y}-2y\dot{x})+2(4A-B)\dot{x}$	$4Dx\dot{x}$	$4L_{xy}\dot{x}+(4Ay+Dx^2+4Dy^2)x^2+\frac{(4A-B)}{D}(\dot{x}^2+Ax^2)$
		3	$B=16A$, $C=-16D$	0	$4\dot{x}^3+4(A+2Dy)x^2\dot{x}-\frac{4}{3}Dx^3\dot{y}$	$-\frac{4}{3}Dx^3\dot{x}$	$\dot{x}^4+2(A+2Dy)x^2\dot{x}^2-\frac{4}{3}Dx^3\dot{x}\dot{y}+A^2x^4-\frac{4}{3}D(A+Dy)x^4y-\frac{2}{9}D^2x^6$
3	Sextic anharmonic oscillator equation (1.10)	1	$A=B$, $\alpha=\beta$, $\delta_1=\delta_2=3\alpha$	0	$2yL_{yx}$	$2xL_{xy}$	L_{xy}^2
		2	$A=B$, $\alpha=\beta$, $\delta_1=\delta_2=15\alpha$	0	$k\dot{y}$	$k\dot{x}$	$\dot{x}\dot{y}+2Axy+6\alpha\rho^4xy+8\alpha x^3y^3$
		3	$A=4B$, $\alpha=64\beta$, $\delta_1=80\beta$, $\delta_2=24\beta$	0	$y\dot{y}$	$y\dot{x}-2x\dot{y}$	$\dot{y}L_{yx}+2(B+3\beta y^4+16\beta\rho^2x^2)xy^2$
4	Perturbed Kepler system equation (1.11)	1	a, b arbitrary, $M=N=-2$	0	yL_{yx}	xL_{xy}	$L_{xy}^2+2b\left(\frac{x}{y}\right)^2+2a\left(\frac{y}{x}\right)^2$
		2	a, b arbitrary, $M=1$, $N=-2$	0	$y\dot{y}$	$y\dot{x}-2x\dot{y}$	$\dot{y}L_{yx}+\frac{gx}{\rho}+\frac{ay^2}{2}-\frac{2bx}{y^2}$

Table 1. (continued)

		3	a, b arbitrary, $M=N=1$	0	$\left[-\frac{2b}{a}y\dot{x}+\left(\frac{b}{a}x+y\right)\dot{y}\right]$	$\left[\left(\frac{b}{a}x+y\right)\dot{x}-2x\dot{y}\right]$	$(-b\dot{x}+a\dot{y})(y\dot{x}-x\dot{y})+\frac{g(ax+by)}{\rho}+\frac{1}{2}(a^2y^2+b^2x^2-2abxy)$
		4	$a=b$, $M=N=2$	0	yL_{yx}	xL_{xy}	L_{xy}
		5	$a=4b$, $M=N=2$	0	$y\dot{y}$	$y\dot{x}-2x\dot{y}$	$\dot{y}L_{yx}+\frac{gx}{\rho}+2bxy^2$
5	Inverse square potential equation (1.12)	1	A, B, C, D arbitrary	0	$yL_{yx}+(B-A)\dot{x}$	xL_{xy}	$L_{yx}^2+2C\left(\frac{y}{x}\right)^2+2D\left(\frac{x}{y}\right)^2+(B-A)\left[\dot{x}^2+2x^4+2x^2y^2+2Ax^2+\frac{2C}{x^2}\right]$
6	Non-homogeneous potential equation (1.13)	1	$B=10A$, $B=2C$, $E=6F$, $H=3G$, $I=J$, $E=6D$	0	$k\dot{y}$	$k\dot{x}$	$\dot{x}\dot{y}+y[A(5x^4+10x^2y^2+y^4)+4D(x^3+xy^2)+G(3x^2+y^2)+2Ix+K]$
		2	$A=B$, $C=\frac{3}{16}A$, $E=\frac{3}{4}D$, $F=\frac{D}{16}$, $H=G/2$, $J=I/4$	0	$y\dot{y}$	$y\dot{x}-2x\dot{y}$	$\dot{y}L_{yx}+\frac{y^2}{2}\left[A\left(x^4+\frac{3}{4}x^2y^2+\frac{y^4}{16}\right)+D(x^3+xy^2/2)+G(x^2+y^2/4)+Ix+K\right]$

$L_{xy}=x\dot{y}-y\dot{x}$, $\rho^2=x^2+y^2$.

(2) Henon–Heiles system [13]:

$$L=(1/2)(\dot{x}^2+\dot{y}^2)-[(1/2)(Ax^2+By^2)+Dx^2y-(1/3)Cy^3]. \tag{1.9}$$

(3) Sextic anharmonic oscillator [14]:

$$L=(1/2)(\dot{x}^2+\dot{y}^2)-[Ax^2+By^2+\alpha x^6+\beta y^6+\delta_1 x^4y^2+\delta_2 x^2y^4]. \tag{1.10}$$

(4) Perturbed Kepler system [15]:

$$L=(1/2)(\dot{x}^2+\dot{y}^2)-\left[\frac{-g}{(x^2+y^2)^{1/2}}+ax^M+by^N\right]. \tag{1.11}$$

(5) Inverse square potential [16]:

$$L=(1/2)(\dot{x}^2+\dot{y}^2)-\left[Ax^2+By^2+(x^2+y^2)^2+\frac{C}{x^2}+\frac{D}{y^2}\right]. \tag{1.12}$$

(6) Non-homogeneous potential [17]:

$$L=(1/2)(\dot{x}^2+\dot{y}^2)-[Ax^5+Bx^3y^2+Cxy^4+Dx^4+Ex^2y^2+Fy^4 \\ +Gx^3+Hxy^2+Ix^2+Jy^2+Kx]. \tag{1.13}$$

In the above $A, B, C, D, E, F, G, H, I, J, K, M, N, \alpha, \beta, \delta, a$ and b are constant parameters.

The plan of the paper is as follows. In section 2 we present the method of constructing the invariants by analysing the associated characteristic equation. As an illustration in section 3 we apply this method to a specific case of the coupled quartic anharmonic oscillator and explicitly demonstrate the method. In section 4 we obtain the separable coordinates, whenever they exist, by integrating a subset of the characteristic equation corresponding to the coordinate variables for the potentials in table 1. In section 5 we give a summary of our results.

2. Method of constructing invariants

Let us consider the one-parameter (ε) Lie group of continuous transformations associated with the infinitesimal generator E of (1.6). Then the local group invariant $U(t, x, y, \dot{x}, \dot{y})$ can be found by solving the first-order linear, homogeneous, partial differential equation [1, 2]

$$E\{U\}=\xi\frac{\partial U}{\partial t}+\eta_1\frac{\partial U}{\partial x}+\eta_2\frac{\partial U}{\partial y}+(\dot{\eta}_1-\dot{x}\dot{\xi})\frac{\partial U}{\partial \dot{x}}+(\dot{\eta}_2-\dot{y}\dot{\xi})\frac{\partial U}{\partial \dot{y}}=0. \tag{2.1}$$

From the classical theory of partial differential equations, we can easily see that equation (2.1) admits four functionally independent solutions or group invariants $U_i(t, x, y, \dot{x}, \dot{y})$, $i=1,2,3,4$. These solutions are the four essential constants which appear in the general solution of the system of four first-order ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\eta_1}{\xi} \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\eta_2}{\xi} \qquad \frac{\mathrm{d}\dot{x}}{\mathrm{d}t}=\frac{(\dot{\eta}_1-\dot{x}\dot{\xi})}{\xi} \qquad \frac{\mathrm{d}\dot{y}}{\mathrm{d}t}=\frac{(\dot{\eta}_2-\dot{y}\dot{\xi})}{\xi}. \tag{2.2}$$

System (2.2) leads to the following characteristic equation [2] associated with (2.1):

$$\frac{\mathrm{d}t}{\xi}=\frac{\mathrm{d}x}{\eta_1}=\frac{\mathrm{d}y}{\eta_2}=\frac{\mathrm{d}\dot{x}}{(\dot{\eta}_1-\dot{x}\dot{\xi})}=\frac{\mathrm{d}\dot{y}}{(\dot{\eta}_2-\dot{y}\dot{\xi})}. \tag{2.3}$$

To find the function U_i, $i=1,2,3,4$ we observe that any tangential direction through a point $(t, x, y, \dot{x}, \dot{y})$ to the surface $U_i(t, x, y, \dot{x}, \dot{y})=C_i$, $i=1,2,3,4$ satisfies the relation [18]

$$\frac{\partial U_i}{\partial t}\,\mathrm{d}t+\frac{\partial U_i}{\partial x}\,\mathrm{d}x+\frac{\partial U_i}{\partial y}\,\mathrm{d}y+\frac{\partial U_i}{\partial \dot{x}}\,\mathrm{d}\dot{x}+\frac{\partial U_i}{\partial \dot{y}}\,\mathrm{d}\dot{y}=0 \qquad i=1,2,3,4. \tag{2.4}$$

If $U_i(t, x, y, \dot{x}, \dot{y})=C_i$ is a suitable one-parameter system of surfaces, the tangential directions to the integral curves of the characteristic equation (2.3) through the point $(t, x, y, \dot{x}, \dot{y})$ are also tangential directions to these surfaces. Hence

$$\xi\frac{\partial U_i}{\partial t}+\eta_1\frac{\partial U_i}{\partial x}+\eta_2\frac{\partial U_i}{\partial y}+(\dot{\eta}_1-\dot{x}\dot{\xi})\frac{\partial U_i}{\partial \dot{x}}+(\dot{\eta}_2-\dot{y}\dot{\xi})\frac{\partial U_i}{\partial \dot{y}}=0 \qquad i=1,2,3,4. \tag{2.5}$$

Then to find U_i $(i=1,2,3,4)$, we try to find functions P_i, Q_i, R_i, S_i and T_i such that

$$\xi P_i+\eta_1 Q_i+\eta_2 R_i+(\dot{\eta}_1-\dot{x}\dot{\xi})S_i+(\dot{\eta}_2-\dot{y}\dot{\xi})T_i=0 \qquad i=1,2,3,4 \tag{2.6}$$

with the property

$$P_i=\frac{\partial U_i}{\partial t} \qquad Q_i=\frac{\partial U_i}{\partial x} \qquad R_i=\frac{\partial U_i}{\partial y} \qquad S_i=\frac{\partial U_i}{\partial \dot{x}} \qquad T_i=\frac{\partial U_i}{\partial \dot{y}} \tag{2.7}$$

so that

$$[P_i\,\mathrm{d}t+Q_i\,\mathrm{d}x+R_i\,\mathrm{d}y+S_i\,\mathrm{d}\dot{x}+T_i\,\mathrm{d}\dot{y}]$$

is an exact differential $\mathrm{d}U_i$.

The interesting point here is that for all the systems with two degrees of freedom considered in this article, see table 1 with $\xi=0$, one can find at least two non-trivial group invariants U_1 and U_2 straightforwardly which turn out to be nothing but the two required involutive integrals of motion. This can be done by choosing the functions P_i, Q_i, R_i, S_i and T_i, $i=1,2$ in (2.6) as follows:

$$\text{(i)} \qquad P_1=0 \qquad Q_1=-\alpha_1=-\frac{\partial L}{\partial x} \qquad R_1=-\alpha_2=-\frac{\partial L}{\partial y}$$
$$S_1=\dot{x}=\frac{\partial L}{\partial \dot{x}} \qquad T_1=\dot{y}=\frac{\partial L}{\partial \dot{y}} \tag{2.8}$$

so that U_1 is the Hamiltonian for all our integrable cases. Similarly

$$\text{(ii)} \qquad P_2=0 \qquad Q_2=-\dot{\eta}_1 \qquad R_2=-\dot{\eta}_2$$
$$S_2=\eta_1 \qquad T_2=\eta_2 \tag{2.9}$$

so that U_2 is the second integral of motion in all the following examples.

The remaining two group invariants U_3 and U_4 can be obtained as follows:

(i) Since the infinitesimal ξ is zero in all our cases, the time part of the characteristic equation (2.3) separates out as $\mathrm{d}t/0$ so that the third invariant is just $U_3=t$.

(ii) Now the remaining part of the characteristic equation

$$\frac{\mathrm{d}x}{\eta_1(x, y, \dot{x}, \dot{y})}=\frac{\mathrm{d}y}{\eta_2(x, y, \dot{x}, \dot{y})}=\frac{\mathrm{d}\dot{x}}{\dot{\eta}_1(x, y, \dot{x}, \dot{y})}=\frac{\mathrm{d}\dot{y}}{\dot{\eta}_2(x, y, \dot{x}, \dot{y})} \tag{2.10}$$

can be reduced to an ordinary differential equation (ODE), say in x and y, by eliminating the other two variables by using the expressions for the first two invariants U_1 and U_2, for example, of the form $\mathrm{d}x/[g(x, y)]=\mathrm{d}y/[h(x, y)]$, which in general is nonlinear.

The solution of this ODE gives the required additional group invariant U_4. For example, for the circularly symmetric cases of a sextic anharmonic oscillator (equation (1.10), case 1) and the perturbed Kepler system (equation (1.11), cases 1 and 4), we have $\mathrm{d}x/y = \mathrm{d}x/-x$, leading to $U_4 = (x^2 + y^2)$. In other cases this first-order ODE cannot in general be explicitly integrated, but in principle its solution is the required group invariant.

3. Application

The method can be straightforwardly applied to any nonlinear oscillator system with two degrees of freedom whose generalized symmetries are known. In particular all the systems (1.8)–(1.13) whose symmetries are given in table 1 can be analysed in this way. As a specific illustration, we treat below case 1 of the coupled quartic anharmonic oscillator system (1.8) explicitly. The two required involutive integrals of motion (including the Hamiltonian) of all the remaining cases and systems can be obtained by following the same steps with relevant symmetries.

Example: Coupled quartic anharmonic oscillator

Case 1.

$$L = (1/2)(\dot{x}^2 + \dot{y}^2) - [Ax^2 + By^2 + \alpha x^4 + \alpha y^4 + 2\alpha x^2 y^2] \qquad A, B \text{ arbitrary} \tag{3.1}$$

Generalized Lie symmetries (see table 1):

$$\begin{gathered} \xi = 0 \qquad \eta_1 = 2y(y\dot{x} - x\dot{y}) + (2/\alpha)(B - A)\dot{x} \qquad \eta_2 = 2x(x\dot{y} - y\dot{x}) \\ \dot{\eta}_1 = 2\dot{y}(\dot{x}y - x\dot{y}) - 4(B - A)xy^2 - 8(B - A)x^3 - (4A/\alpha)(B - A)x \\ \dot{\eta}_2 = 2\dot{x}(x\dot{y} - y\dot{x}) + 4(A - B)x^2 y. \end{gathered} \tag{3.2}$$

The characteristic equation is

$$\begin{aligned} \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{2y(y\dot{x} - x\dot{y}) + (2/\alpha)(B - A)\dot{x}} &= \frac{\mathrm{d}y}{2x(x\dot{y} - y\dot{x})} \\ &= \frac{\mathrm{d}\dot{x}}{2\dot{y}(y\dot{x} - x\dot{y}) - 4(B - A)[xy^2 + 2x^3 + (A/\alpha)x]} \\ &= \frac{\mathrm{d}\dot{y}}{2\dot{x}(x\dot{y} - y\dot{x}) + 4(A - B)x^2 y}. \end{aligned} \tag{3.3}$$

(a) First invariant. We choose

$$P_1 = 0 \qquad Q_1 = -\frac{\partial L}{\partial x} = (2Ax + 4\alpha x^3 + 4\alpha x y^2)$$

$$R_1 = -\frac{\partial L}{\partial y} = (2By + 4\alpha y^3 + 4\alpha x^2 y) \qquad S_1 = \frac{\partial L}{\partial \dot{x}} = \dot{x} \qquad T_1 = \frac{\partial L}{\partial \dot{y}} = \dot{y} \tag{3.4}$$

so that

$$\xi P_1 + \eta_1 Q_1 + \eta_2 R_1 + (\dot{\eta}_1 - \dot{x}\dot{\xi}) S_1 + (\dot{\eta}_2 - \dot{y}\dot{\xi}) T_1 = 0 \tag{3.5}$$

and

$$\frac{\partial U_1}{\partial t}=0 \qquad \frac{\partial U_1}{\partial x}=(2Ax+4\alpha x^3+4\alpha xy^2) \qquad \frac{\partial U_1}{\partial y}=(2By+4\alpha y^3+4\alpha x^2 y)$$
$$\frac{\partial U_1}{\partial \dot{x}}=\dot{x} \qquad \frac{\partial U_1}{\partial \dot{y}}=\dot{y}. \tag{3.6}$$

Integrating (3.6) we obtain the invariant

$$U_1=(1/2)(\dot{x}^2+\dot{y}^2)+(Ax^2+By^2+\alpha x^4+\alpha y^4+2\alpha x^2y^2) \tag{3.7}$$

which is nothing but the Hamiltonian of (3.1).
(b) Second invariant. Now we choose

$$P_2=0 \qquad Q_2=-\dot{\eta}_1=2\dot{y}(x\dot{y}-y\dot{x})+(4A/\alpha)(B-A)x+8(B-A)x^3+4(B-A)xy^2$$
$$R_2=-\dot{\eta}_2=2\dot{x}(y\dot{x}-x\dot{y})+4(B-A)x^2y$$
$$S_2=\eta_1=2y(y\dot{x}-x\dot{y})+(2/\alpha)(B-A)\dot{x} \qquad T_2=\eta_2=2x(x\dot{y}-y\dot{x}) \tag{3.8}$$

so that

$$\xi P_2+\eta_1 Q_2+\eta_2 R_2+(\dot{\eta}_1-\dot{x}\dot{\xi})S_2+(\dot{\eta}_2-\dot{y}\dot{\xi})T_2=0 \tag{3.9}$$

and

$$\frac{\partial U_2}{\partial t}=0 \qquad \frac{\partial U_2}{\partial x}=2\dot{y}(x\dot{y}-y\dot{x})+(4A/\alpha)(B-A)x+8(B-A)x^3+4(B-A)xy^2$$
$$\frac{\partial U_2}{\partial y}=2\dot{x}(y\dot{x}-x\dot{y})+4(B-A)x^2y \qquad \frac{\partial U_2}{\partial \dot{x}}=2y(y\dot{x}-x\dot{y})+(2/\alpha)(B-A)\dot{x} \tag{3.10}$$
$$\frac{\partial U_2}{\partial \dot{y}}=2x(x\dot{y}-y\dot{x}).$$

Integrating (3.10) we get

$$U_2=I_2=(x\dot{y}-y\dot{x})^2+(2/\alpha)(B-A)[(\dot{x}^2/2)+Ax^2+\alpha x^4+\alpha x^2y^2] \tag{3.11}$$

which is nothing but the required second involutive integral of motion given in table 1.

As noted in section 2, the third invariant is $U_3=t$. Further, from the expressions (3.7) and (3.11) for U_1 and U_2 respectively, two of the variables say $\dot{x}$ and $\dot{y}$ can be expressed as functions of x and y so that the characteristic equation (3.3) can be expressed as a highly complicated nonlinear first-order ODE, whose solution is the required invariant U_4. However, for the circularly symmetric case $A=B$, the characteristic equation (3.3) degenerates into

$$\frac{dx}{y}=\frac{dy}{-x}$$

so that we have the explicit functional form $U_4=(x^2+y^2)$.

Similar analysis can be carried out to find the invariants for each of the cases in table 1 and to obtain the Hamiltonian and the second involutive integral of motion I_2 in the required form. Thus the integrability of all these systems can be proved.

4. Separability

Next we will show that from the generalized symmetries, we can also find suitable coordinate systems in which either the equations of motion or the Hamilton–Jacobi equation becomes separable, whenever the generalized symmetries are linear in the velocities. Since the separability is associated with coordinate transformations, we will consider that part of the characteristic equation discussed earlier, namely

$$\frac{\mathrm{d}x}{\eta_1}=\frac{\mathrm{d}y}{\eta_2}. \tag{4.1}$$

When η_1 and η_2 are linear in the velocities, the system (4.1) degenerates into an ODE which can always be integrated to find the suitable separable coordinates. However when η_1 and η_2 are of higher degree in the velocities then equation (4.1) cannot be solved and so no separable coordinates set can be found in general through this procedure.

4.1. *Two coupled quartic anharmonic oscillator*

Case 1. From equation (3.3), we have

$$\frac{\mathrm{d}x}{2y(\dot{x}y-\dot{y}x)+2c^2\dot{x}}=\frac{\mathrm{d}y}{2x(x\dot{y}-y\dot{x})} \qquad \text{where } c^2=\frac{(B-A)}{\alpha}. \tag{4.2}$$

This can be rewritten as

$$xy\left[\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-1\right]+(x^2-y^2-c^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0 \tag{4.3}$$

which can be readily integrated to give

$$(m+1)(mx^2-y^2)-mc^2=0 \tag{4.4}$$

where m is an arbitrary constant. Rewriting (4.4) in the elliptical form

$$\frac{x^2}{\xi^2}+\frac{y^2}{\xi^2-c^2}=1 \qquad \xi^2=\frac{c^2}{m+1} \tag{4.5}$$

we have the obvious parametrization

$$x=\frac{\xi\eta}{c} \qquad y=\frac{1}{c}\left[(\xi^2-c^2)(c^2-\eta^2)\right]^{1/2}. \tag{4.6}$$

Under this transformation the Hamilton–Jacobi equation

$$\begin{aligned}(1/2)\left[\frac{1}{(\xi^2-\eta^2)}\right][(\xi^2-c^2)S_\xi^2+(c^2-\eta^2)S_\eta^2+2\alpha(\xi^6-\eta^6)&\\ +2(2A-B)(\xi^4-\eta^4)-2c^2A(\xi^2-\eta^2)]=E&\end{aligned} \tag{4.7}$$

is separable [19].

Case 2. From table 1 we have

$$\frac{\mathrm{d}x}{k\dot{y}}=\frac{\mathrm{d}y}{k\dot{x}} \tag{4.8}$$

or

$$dx^2 - dy^2 = 0 \tag{4.9}$$

so that

$$x^2 - y^2 = \text{constant} \tag{4.10}$$

leading to the linearly transformed coordinates

$$u = x + y \qquad v = x - y. \tag{4.11}$$

Under this transformation the equation of motion decouples into

$$\frac{d^2u}{dt^2} + 2Au + 4\alpha u^3 = 0 \qquad \frac{d^2v}{dt^2} + 2Av + 4\alpha v^3 = 0 \tag{4.12}$$

which can be solved in terms of the Jacobian elliptic function.

Case 3. From table 1 we have

$$\frac{dx}{y\dot{y}} = \frac{dy}{y\dot{x} - 2x\dot{y}} \tag{4.13}$$

which can be written as

$$y(dy^2 - dx^2) + 2x\, dx\, dy = 0 \tag{4.14}$$

or

$$y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) - y = 0. \tag{4.15}$$

This can be integrated to give

$$y^2 - 2mx - m^2 = 0 \qquad m = \text{constant}. \tag{4.16}$$

Rewriting (4.16), we have

$$\frac{y^2}{\eta^4} - \frac{2x}{\eta^2} = 1 \qquad \eta^2 = m. \tag{4.17}$$

Naturally (4.17) can be parametrized in terms of the parabolic coordinates

$$x = (1/2)(\xi^2 - \eta^2) \qquad y = \xi\eta \tag{4.18}$$

with which the Hamilton–Jacobi equation becomes separable [20].

Case 4. In this case, from table 1 we have

$$\frac{dx}{8\beta(y\dot{x} - 2x\dot{y})y^3} = \frac{dy}{4\dot{y}^3 + 8[B + \beta y^2 + 6\beta x^2]y^2\dot{y} - 16\beta x y^3 \dot{x}} \tag{4.19}$$

or

$$\frac{dy}{dx} = \frac{4\dot{y}^3 + 8[B + \beta y^2 + 6\beta x^2]y^2\dot{y} - 16\beta x y^3 \dot{x}}{8\beta(y\dot{x} - 2x\dot{y})y^3}. \tag{4.20}$$

Due to the $\dot{y}^3$ term in the right-hand side of the above, equation (4.20) cannot be reduced to an ODE as such and so no separable coordinates can be found in this case. One can even make use of the form of H and I_2 to eliminate $\dot{x}$ and $\dot{y}$, and reduce (4.20) to an ODE in x and y, but such a system is too complicated to be of any practical use.

4.2. The Henon–Heiles system

Case 1. From table 1 we have

$$\frac{\mathrm{d}x}{k\dot{y}}=\frac{\mathrm{d}y}{k\dot{x}} \tag{4.21}$$

which is the same as (4.8). So we obtain the known linear transformation (4.11) under which the equaton of motion becomes separable.

Case 2. Now we have

$$\frac{\mathrm{d}x}{4Dx\dot{y}-8Dy\dot{x}+8A\dot{x}-2B\dot{x}}=\frac{\mathrm{d}y}{4Dx\dot{x}} \tag{4.22}$$

so that

$$x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+(k-2y)\frac{\mathrm{d}y}{\mathrm{d}x}-x=0 \qquad k=\frac{(4A-B)}{2D}. \tag{4.23}$$

On integrating (4.23) we have

$$2y-k=mx^2-(1/m) \tag{4.24}$$

where m is an arbitrary constant, which can be parametrized in terms of the shifted parabolic coordinates.

$$x=(\xi\eta)^{1/2} \qquad y=(1/2)(\xi-\eta)+\frac{(4A-B)}{D} \tag{4.25}$$

with which the Hamilton–Jacobi equation becomes separable [13].

Case 3. In this case from table 1 we have

$$\frac{\mathrm{d}x}{4\dot{x}^3+4(A+2Dy)x^2\dot{x}-(4/3)Dx^3\dot{y}}=\frac{\mathrm{d}y}{-(4/3)Dx^3\dot{x}} \tag{4.26}$$

which we are unable to integrate as such and no separable coordinates can be found in this case.

4.3. Sextic anharmonic oscillator

Case 1. In this case, we have

$$\frac{\mathrm{d}x}{2y(y\dot{x}-x\dot{y})}=\frac{\mathrm{d}y}{2x(x\dot{y}-y\dot{x})}. \tag{4.27}$$

This can be rewritten as

$$x\,\mathrm{d}x+y\,\mathrm{d}y=0 \tag{4.28}$$

which can be integrated to give

$$x^2+y^2=\text{constant}. \tag{4.29}$$

In this case we can transform the Cartesian coordinates x and y into polar coordinates $x=\rho\cos\theta$, $y=\rho\sin\theta$, in which case the equation of motion becomes separable [21].

Case 2. From table 1 we have

$$\frac{\mathrm{d}x}{k\dot{y}}=\frac{\mathrm{d}y}{k\dot{x}} \tag{4.30}$$

which readily gives the linear transformation $u=x+y$, $v=x-y$. Under this transformation the equation of motion is itself separable.

Case 3. From table 1 we have

$$\frac{\mathrm{d}x}{y\dot{y}}=\frac{\mathrm{d}y}{y\dot{x}-2x\dot{y}}. \tag{4.31}$$

Integrating (4.31), we get the parabolic cylindrical coordinates $y=\xi\eta$, $x=(1/2)(\xi^2-\eta^2)$ in which the Hamilton–Jacobi equation becomes separable as in the case of the quartic oscillator [19].

4.4. *Perturbed Kepler system*

Case 1. In this case, we have

$$\frac{\mathrm{d}x}{y(y\dot{x}-x\dot{y})}=\frac{\mathrm{d}y}{x(x\dot{y}-y\dot{x})}. \tag{4.32}$$

Rearranging and integrating we get the following polar coordinates $x=\rho\cos\theta$, $y=\rho\sin\theta$, in which the equation of motion becomes separable [21].

Case 2. In this case, we have

$$\frac{\mathrm{d}x}{y\dot{y}}=\frac{\mathrm{d}y}{y\dot{x}-2x\dot{y}}. \tag{4.33}$$

Integrating (4.33), we get the parabolic cylindrical coordinates $y=\xi\eta$, $x=(1/2)(\xi^2-\eta^2)$, in which the Hamilton–Jacobi equation becomes separable.

Case 3. From table 1 we have

$$\frac{\mathrm{d}x}{\{-(2b/a)y\dot{x}+[(b/a)x+y]\dot{y}\}}=\frac{\mathrm{d}y}{\{[(b/a)x+y]\dot{x}-2x\dot{y}\}}. \tag{4.34}$$

Solving (4.34) we get the parabolic coordinates under which the Hamilton–Jacobi equation becomes separable [15].

Case 4. From table 1 we have

$$\frac{\mathrm{d}x}{y(y\dot{x}-x\dot{y})}=\frac{\mathrm{d}y}{x(x\dot{y}-y\dot{x})}. \tag{4.35}$$

Rearranging and integrating we get the polar coordinates $x=\rho\cos\theta$, $y=\rho\sin\theta$, in which the equation of motion becomes separable as (4.32) [15].

Case 5. From table 1 we have

$$\frac{\mathrm{d}x}{y\dot{y}}=\frac{\mathrm{d}y}{y\dot{x}-2x\dot{y}}. \tag{4.36}$$

Rearranging and integrating (4.36) we get the parabolic coordinates in which the Hamilton–Jacobi equation becomes separable as in (4.33) [15, 22].

4.5. *Inverse square potential*

In this case, we have

$$\frac{\mathrm{d}x}{y(y\dot{x}-x\dot{y})+(B-A)\dot{x}}=\frac{\mathrm{d}y}{x(x\dot{y}-y\dot{x})}. \tag{4.37}$$

Rearranging and integrating we get the elliptical coordinates as noted in equation (4.6) under which the Hamilton–Jacobi equation becomes separable [16].

4.6. *Non-homogeneous potential*

Case 1. From table 1 we have

$$\frac{\mathrm{d}x}{k\dot{y}}=\frac{\mathrm{d}y}{k\dot{x}}. \tag{4.38}$$

Rearranging and integrating, we get the linear transformation under which the equation of motion itself becomes separable.

Case 2. In this we have

$$\frac{\mathrm{d}x}{y\dot{y}}=\frac{\mathrm{d}y}{y\dot{x}-2x\dot{y}}. \tag{4.39}$$

Rearranging and integrating we get parabolic coordinates in which the Hamilton–Jacobi equation becomes separable.

5. Conclusions

In this paper we have demonstrated explicitly how to find the integrals of motion associated with a given set of dynamical symmetries for nonlinear dynamical systems with two degrees of freedom obtained through invariance analysis by direct integration. We have also shown that separable coordinates, if they exist, can also be easily found by integrating a subset of the symmetries. Thus one can obtain the required integrability and separability properties directly through invariance analysis.

This method can also be extended to 3D systems in a straightforward manner. In general there exist two sets of non-trivial Lie symmetries [23] here. While the first and second integrals of motion can be obtained in the same way as in section 2, suitably extended to three degrees of freedom, the third integral of motion can be found as in (2.9) but with the second set of symmetries. The details are presented elsewhere [24].

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